OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 5 NOVEMBER 2008

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

NAME: MODEL ANSWERS

TEST CENTRE:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ______________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A–J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The function

$$y = 2x^3 - 6x^2 + 5x - 7$$

has

(a) no stationary points;
(b) one stationary point;
(c) two stationary points;
(d) three stationary points.

$y = 2x^3 - 6x^2 + 5x - 7$

$\frac{dy}{dx} = 6x^2 - 12x + 5$

Stationary points occur when $\frac{dy}{dx} = 0$. To obtain the number of solutions to $\frac{dy}{dx} = 0$ we can use the discriminant: $b^2 - 4ac$

$= (-12)^2 - 4(6)(5)$

$= 144 - 120$

$= 24 > 0$

So there are two solutions to $\frac{dy}{dx} = 0$ and hence two stationary points.

Recall:
$b^2 - 4ac = 0$ repeated root
$b^2 - 4ac > 0$ two distinct roots
$b^2 - 4ac < 0$ no real roots

B. Which is the smallest of these values?

(a) $\log_{10} \pi$, (b) $\sqrt{\log_{10}(\pi^2)}$, (c) $\left(\frac{1}{\log_{10} \pi}\right)^3$, (d) $\frac{1}{\log_{10} \sqrt{\pi}}$.

since the value of π is less than 10,

$L = \log_{10} \pi < 1$ define it to make it easy to work with

(a) $\log_{10} \pi = L$

(b) $\sqrt{\log_{10}(\pi^2)} = \sqrt{2 \log_{10} \pi}$

$= \sqrt{2L} > \sqrt{L \times L} = L$, since we know $L < 1$, we have $L < 2$.

$\Rightarrow \sqrt{2L} > L$

(c) $\left(\frac{1}{\log_{10} \pi}\right)^3 = L^{-3} > 1 > L$

(d) $\frac{1}{\log_{10} \sqrt{\pi}} = \frac{1}{\log_{10} (\pi)^{\frac{1}{2}}}$

$= \frac{1}{\frac{1}{2} \log_{10} \pi}$

$= \frac{2}{L} > 2 > L.$

C. The simultaneous equations in x, y,

$$(\cos\theta)\,x - (\sin\theta)\,y = 2 \quad — ①$$
$$(\sin\theta)\,x + (\cos\theta)\,y = 1 \quad — ②$$

are solvable

(a) for all values of θ in the range $0 \leqslant \theta < 2\pi$;
(b) except for one value of θ in the range $0 \leqslant \theta < 2\pi$;
(c) except for two values of θ in the range $0 \leqslant \theta < 2\pi$;
(d) except for three values of θ in the range $0 \leqslant \theta < 2\pi$.

Set $\cos\theta = c$ and $\sin\theta = s$ for ease of notation:

$cx - sy = 2$ ①
$sx + cy = 1$ ②

Eliminate y by $c\times① + s\times②$: $c^2x + s^2x = 2c + s$
$\Rightarrow x = 2c + s$

Eliminate x by $c\times② - s\times①$: $c^2y + s^2y = c - 2s$
$\Rightarrow y = c - 2s$

here we've used $c^2 + s^2 = 1$ since $\cos^2\theta + \sin^2\theta = 1$

The equations $x = 2\cos\theta + \sin\theta$, $y = \cos\theta - 2\sin\theta$ are clearly solvable for all values of θ in the range $0 \leq \theta < 2\pi$, i.e. substituting any value of θ in $[0, 2\pi)$ will give a valid solution for x and y.

D. When

$$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

is divided by $x - 1$ the remainder is

(a) 2000, (b) 2500, (c) 3000, (d) 3500.

Remainder theorem: When a polynomial $p(x)$ is divided by $(x-1)$, the remainder is $p(1)$.

Let $p(x) = 1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$

Then

$$p(1) = 1 + 3 + 5 + 7 + \cdots + 99$$
$$= \frac{50}{2}(1 + 99)$$
$$= 25(100) = 2500 \;//$$

Here we've used the formula for the sum of finite arithmetic progression:

$S_n = \frac{n}{2}(a_1 + a_n)$,

where $n = 50$, $a_1 = 1$, $a_n = 99$

E. The highest power of x in

$$\left\{\left[\left(2x^6+7\right)^3+\left(3x^8-12\right)^4\right]^5+\left[\left(3x^5-12x^2\right)^5+\left(x^7+6\right)^4\right]^6\right\}^3$$

is

(a) x^{424}, (b) x^{450}, (c) x^{500}, (d) x^{504}.

$$\Bigg\{\underbrace{\Big[\underbrace{(2x^6+7)^3}_{x^{18}}+\underbrace{(3x^8-12)^4}_{x^{32}}\Big]^5}_{x^{160}}+\underbrace{\Big[\underbrace{(3x^5-12x^2)^5}_{x^{25}}+\underbrace{(x^7+6)^4}_{x^{28}}\Big]^6}_{x^{168}}\Bigg\}^3$$

$$\underbrace{\qquad\qquad\qquad}_{x^{504}}$$

x^{504} ← comes from $(x^{168})^3 = x^{504}$

F. If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,\mathrm{d}x,$$

by splitting the interval $0 \leqslant x \leqslant 1$ into 10 intervals then an **overestimate** of the integral is produced. It follows that

(a) the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\,\mathrm{d}x$;

(b) the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x)-1)\,\mathrm{d}x$;

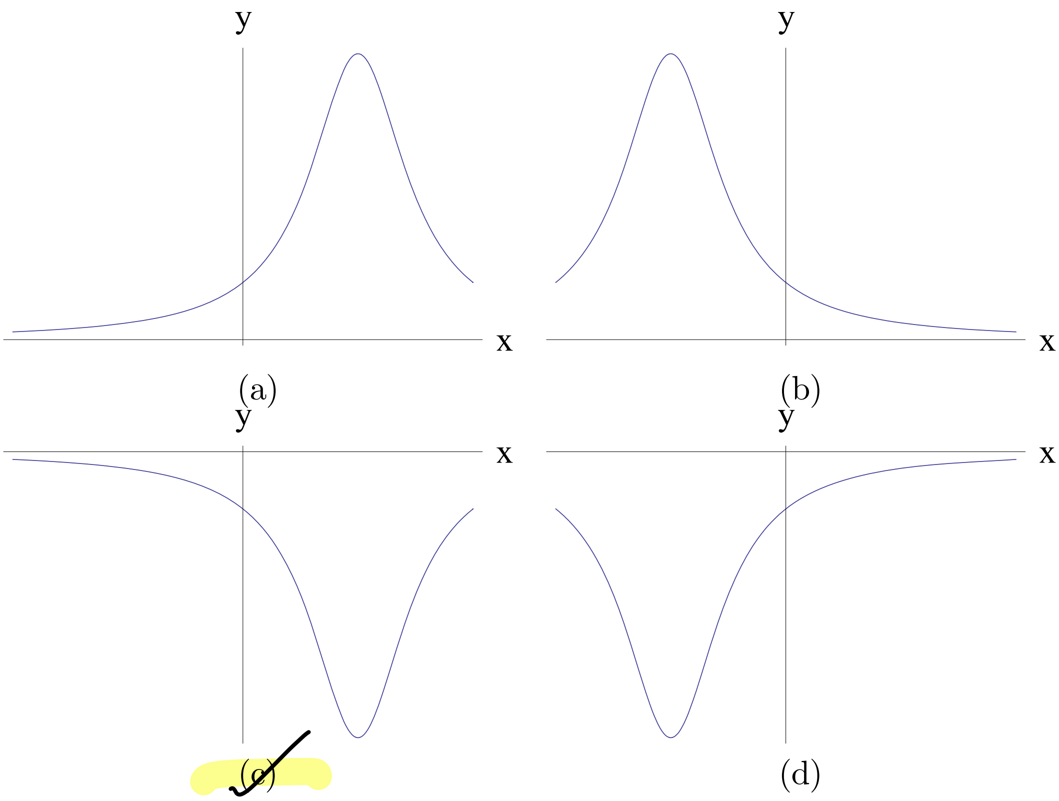
(c) the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\,\mathrm{d}x$;

(d) the trapezium rule with 10 intervals underestimates $\int_0^1 (1-f(x))\,\mathrm{d}x$.

$\int_0^1 (1-f(x))dx$ is the area under the reflection of $f(x)$ in the x axis, followed by a translation of 1 unit parallel to the y axis. The reflection in the x axis changes the overestimate to an underestimate, while the translation does not affect it.

G. Which of the graphs below is a sketch of

$$y = \frac{1}{4x - x^2 - 5} \quad ?$$



$$y = \frac{1}{4x - x^2 - 5}$$

$x = 0$

$y = -\frac{1}{5}$

eliminating a and b

$$\frac{dy}{dx} = -1\ (4x - x^2 - 5)^{-2} \times (4 - 2x) = 0$$

$4 - 2x = 0$

$x = 0$

turning point at $x = 2$,

eliminating d

$\therefore$ answer is c

H. The equation

$$9^x - 3^{x+1} = k$$

has one or more real solutions precisely when

(a) $k \geqslant -9/4$, (b) $k > 0$, (c) $k \leqslant -1$, (d) $k \geqslant 5/8$.

$9^x - 3^{x+1} = k$

$3^{2x} - 3 \times 3^x - k = 0$

let $y = 3^x$ $\quad y^2 - 3y - k = 0$

$y = \dfrac{3 \pm \sqrt{9+4k}}{2}$ $\quad \therefore$ has solutions for $k \geqslant -\frac{9}{4}$

as $y = 3^x$

$y \geqslant 0$ but larger root always positive

I. The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7 + 2 + 3 = 12$. The sum

$$S(1) + S(2) + S(3) + \cdots + S(99)$$

equals

(a) 746, (b) 862, (c) 900, (d) 924.

$S(1) + S(2) + \ldots + S(99)$

The first digit includes ten of each number from 1 to 9.

The second digit includes ten of each number from 0 to 9.

$\therefore$ the sum $= 20 \times (1 + 2 + \ldots + 9)$

$= 20 \times 45$

$= 900$

J. In the range $0 \leqslant x < 2\pi$ the equation

$$(3 + \cos x)^2 = 4 - 2\sin^8 x$$

has

(a) 0 solutions, (b) 1 solution, (c) 2 solutions, (d) 3 solutions.

$-1 \leq \cos x \leq 1$

$2 \leq (3 + \cos x) \leq 4$

$4 \leq (3 + \cos x)^2 \leq 16$

$-1 \leq \sin x \leq 1$

$0 \leq \sin^8 x \leq 1$

$-2 \leq -2\sin^8 x \leq 0$

$2 \leq 4 - 2\sin^8 x \leq 4$

The only point $(3+\cos x)^2 = 4 - 2\sin^8 x$ has a solution is when $(3+\cos x)^2 = 4 - 2\sin^8 x = 4$ and $\therefore$ there is 1 solution

2. For **ALL APPLICANTS.**

(i) Find a pair of positive integers, x_1 and y_1, that solve the equation

$$(x_1)^2 - 2(y_1)^2 = 1.$$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \qquad y_{n+1} = ax_n + by_n, \qquad \text{for } n \geqslant 1.$$

Find *positive* values for a, b such that

$$(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2.$$

(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

(iv) (Using the values of a and b found in part (ii)) what is the approximate value of x_n/y_n as n increases?

i) $x_1 = 3$, $y_1 = 2$ (trial & error)

ii) $(3x_n + 4y_n)^2 - 2(by_n + ax_n)^2 = x_n^2 - 2y_n^2$

$9x_n^2 + 16y_n^2 + 24x_ny_n - 2(a^2x_n^2 + b^2y_n^2 + 2aby_nx_n) = x_n^2 - 2y_n^2$

$9x_n^2 - 2a^2x_n^2 = x_n^2$

$8 = 2a^2$

$a^2 = 4$ $(a>0)$

$a = 2$

$16y_n^2 - 2b^2y_n^2 = -2y_n^2$

$18 = 2b^2$

$b^2 = 9$ $(b>0)$

$b = 3$

$24x_ny_n - 4abx_ny_n = 0$

$24 = 4ab$

sub in $a=2$, $b=3$ to check:

$4 \times 3 \times 2 = 24$

iii) $x_{n+1} = 3x_n + 4y_n$ $\quad y_{n+1} = 2x_n + 3y_n$

$x_1 = 3$, $y_1 = 2$

$x_2 = 3\times3 + 4\times2$

$x_2 = 17$

$y_2 = 2\times3 + 3\times2$

$= 12$

$(17 < 50)$

$x_3 = 3\times17 + 4\times12$

$x_3 = 99$

$y_3 = 2\times17 + 3\times12$

$= 70$

$99 > 70 > 50$

$\therefore X = 99 \quad Y = 70$

iv) $x_n^2 - 2y_n^2 = 1$

$$\frac{x_n^2}{y_n^2} - \frac{2y_n^2}{y_n^2} = \frac{1}{y_n^2}$$

$$\left(\frac{x_n}{y_n}\right)^2 - 2 = \frac{1}{y_n^2}$$

as y_n^2 increases, $\frac{1}{y_n^2} \to 0$

$$\left(\frac{x_n}{y_n}\right)^2 = 2$$

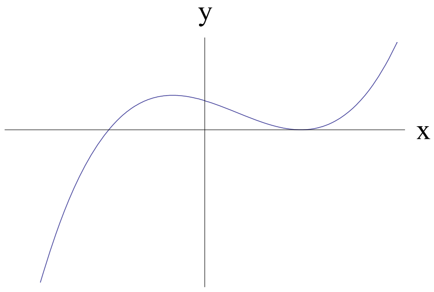
$$\frac{x_n}{y_n} = \sqrt{2}$$

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

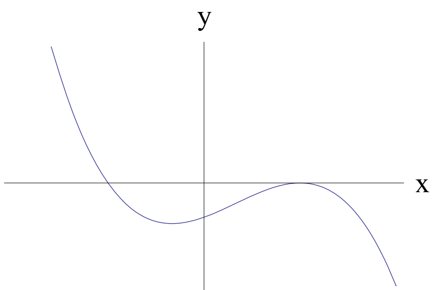
(i) The graph $y = f(x)$ of a certain function has been plotted below.



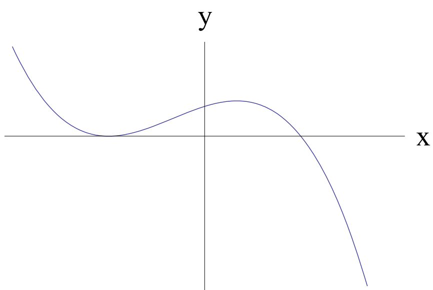
On the next three pairs of axes (A), (B), (C) are graphs of

$$y = f(-x), \qquad f(x-1), \qquad -f(x)$$

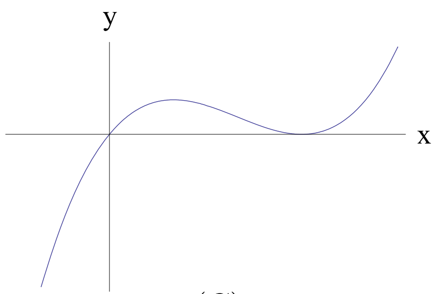
in some order. Say which axes correspond to which graphs.



(A)



(B)



(C)

(ii) Sketch, on the axes opposite, graphs of *both* of the following functions

$$y = 2^{-x^2} \qquad \text{and} \qquad y = 2^{2x-x^2}.$$

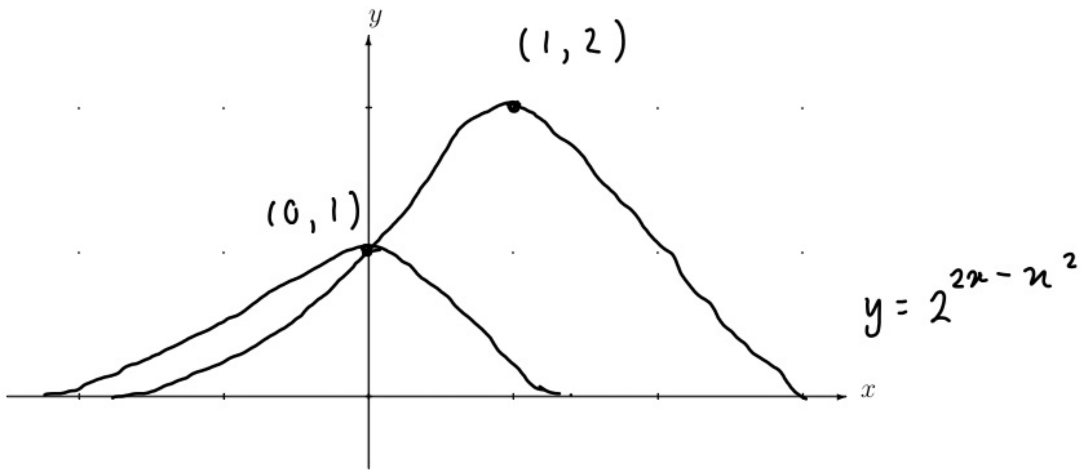
Carefully label any stationary points.

(iii) Let c be a real number and define the following integral

$$I(c) = \int_0^1 2^{-(x-c)^2}\, dx.$$

State the value(s) of c for which $I(c)$ is largest. Briefly explain your reasoning.
[Note you are not being asked to calculate this maximum value.]

ii)



i) $y = f(-x)$, reflection in y axis, B

$y = -f(x)$, reflection in x axis, A

$y = f(x-1)$, translation of $\binom{1}{0}$, C

ii) $y = 2^{-x^2} = e^{-x^2 \ln 2}$

$\frac{dy}{dx} = -2x(\ln 2) 2^{-x^2} = 0$

$2^{-x^2} \neq 0 \quad x = 0$

$2x - x^2 = -[x^2 - 2x]$

$= -[(x-1)^2 - 1]$

$= 1 - (x-1)^2$

$y = 2^{2x - x^2} = 2^{1-(x-1)^2}$

$= 2 \times 2^{(x-1)^2}$

$y = 2^{2x-x^2}$ is $y = 2^{-x^2}$ translated by $\binom{1}{0}$ followed by a stretch parallel to the y axis with scale factor 2.

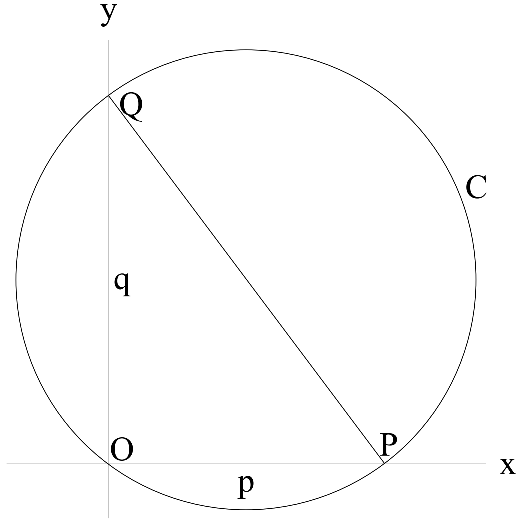
$\therefore$ turning point of $y = 2^{2x-x^2}$ is at $(1, 2)$

iii) $c = \frac{1}{2}$ so that the highest part of $y = 2^{-x^2}$ is moved to the middle of 0 and 1, $\therefore$ maximising the area underneath the graph and maximising $I(c)$.

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**



Let p and q be positive real numbers. Let P denote the point $(p, 0)$ and Q denote the point $(0, q)$.

(i) Show that the equation of the circle C which passes through P, Q and the origin O is

$$x^2 - px + y^2 - qy = 0.$$

Find the centre and area of C.

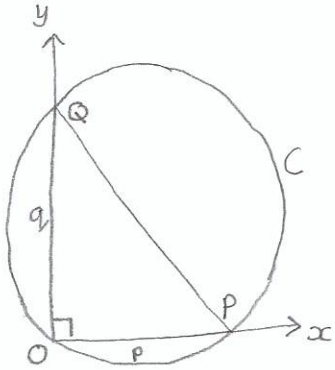
(ii) Show that

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} \geqslant \pi.$$

(iii) Find the angles OPQ and OQP if

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = 2\pi.$$

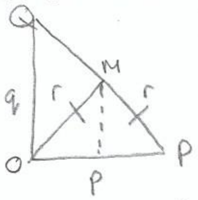
4.



i. As $\angle QOP = 90°$, QP must be the diameter of the circle

Let M be the centre of the circle. $OM = PM = r$

$\therefore M\left(\frac{p}{2}, \frac{q}{2}\right)$



$q^2 + p^2 = (QP)^2 \qquad QP = 2r$

$\frac{q^2+p^2}{4} = r^2$

$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{q}{2}\right)^2 = \frac{q^2+p^2}{4}$

$x^2 - px + \frac{p^2}{4} + y - qy + \frac{q^2}{4} = \frac{q^2+p^2}{4}$

$x^2 - px + y^2 - qy = 0$

Centre of circle $\left(\frac{p}{2}, \frac{q}{2}\right)$

Area of circle $= \pi r^2 = \pi\left(\frac{q^2+p^2}{4}\right)$

ii. Area of circle $C = \pi\left(\frac{q^2+p^2}{4}\right)$

Area of triangle $OPQ = \frac{pq}{2}$

$\pi\left(\frac{q^2+p^2}{4}\right) \div \frac{pq}{2} = \pi\left(\frac{q^2+p^2}{2pq}\right)$

$\pi\left(\frac{p^2+q^2}{2pq}\right) \geq \pi$

$p^2 - q^2 \geqslant 2pq$

$p^2 - q^2 - 2pq \geqslant 0$

$(p-q)^2 \geqslant 0$ $\therefore$ true as squaring $(p-q)$ means it is always positive

iii. Let angle $OPQ = \alpha$

$\tan\alpha = \frac{q}{p}$

$\pi\left(\frac{p^2+q^2}{2pq}\right) = 2\pi$

$p^2 + q^2 = 4pq$

$\frac{p^2}{q^2} + 1 = \frac{4p}{q}$

$\frac{p^2}{q^2} - \frac{4p}{q} + 1 = 0$

$\frac{p}{q} = \frac{4 \pm \sqrt{4^2 - 4\times1\times1}}{2}$

$\frac{p}{q} = \frac{4 \pm \sqrt{12}}{2}$

$\frac{p}{q} = 2 \pm \sqrt{3}$

Let angle $OQP = \beta$

$\tan\beta = \frac{p}{q}$

$\tan\beta = 2 + \sqrt{3} \qquad \tan\alpha = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$

5. For **ALL APPLICANTS.**

The Millennium school has 1000 students and 1000 student lockers. The lockers are in a line in a long corridor and are numbered from 1 to 1000.

Initially all the lockers are closed (but unlocked).

The first student walks along the corridor and opens every locker.

The second student then walks along the corridor and closes every second locker, i.e. closes lockers 2, 4, 6, etc. At that point there are 500 lockers that are open and 500 that are closed.

The third student then walks along the corridor, changing the state of every third locker. Thus s/he closes locker 3 (which had been left open by the first student), opens locker 6 (closed by the second student), closes locker 9, etc.

All the remaining students now walk by in order, with the kth student changing the state of every kth locker, and this continues until all 1000 students have walked along the corridor.

(i) How many lockers are closed immediately after the third student has walked along the corridor? Explain your reasoning.

(ii) How many lockers are closed immediately after the fourth student has walked along the corridor? Explain your reasoning.

(iii) At the end (after all 1000 students have passed), what is the state of locker 100? Explain your reasoning.

(iv) After the *hundredth* student has walked along the corridor, what is the state of locker 1000? Explain your reasoning.

5i. Intially 1000 lockers are closed.
After 1st person, 1000 lockers are open.
After 2nd person, 500 open (odd numbers), 500 closed (even numbers).
After 3rd person, 333 lockers have changed state. 167 lockers affected are odd (and become closed), 166 are even (and become open). 333 (odd) lockers remain open, 334 (even) remain closed.
∴ 501 lockers are closed, 499 are open.

ii. After 4th person, 250 lockers have changed state. All affected are even. 83 of the 166 even open lockers are divisible by 4 and become closed. 167 of 334 even closed lockers are divisible by 4 and become open. 334 lockers remain closed, 416 remain open.
∴ 417 lockers are closed, 583 are open.

iii.

Factors of 100		1	2	4	5	10	20	25	50	100
State of Locker 100	C	O	C	O	C	O	C	O	C	O

∴ Locker 100 is open at the end

iv.

Factors of 1000 between 1 and 100		1	2	4	5	8	10	20	25	40	50	100
State of Locker 1000	C	O	C	O	C	O	C	O	C	O	C	O

∴ Locker 1000 is open after the 100th student

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

(i) A, B and C are three people. One of them is a *liar* who always tells lies, another is a *saint* who always tells the truth, and the third is a *switcher* who sometimes tells the truth and sometimes lies. They make the following statements:

A: I am the liar.
B: A is the liar.
C: I am not the liar.

Who is the liar among A, B and C? Briefly explain your answer.

(ii) P, Q and R are three more people, one a liar, one a saint, and the third a *contrarian* who tells a lie if he is the first to speak or if the preceding speaker told the truth, but otherwise tells the truth. They make the following statements:

P: Q is the liar.
Q: R is the liar.
R: P is the liar.

Who is the liar among P, Q and R? Briefly explain your answer.

(iii) X, Y and Z are three more people, one a liar, one a switcher and one a contrarian. They make the following statements:

X: Y is the liar.
Y: Z is the liar.
Z: X is the liar.
X: Y is the liar.
Y: X is the liar.

Who is the liar among X, Y and Z? Briefly explain your answer.

6i. The statement "I am the liar" cannot be said by a liar or a saint. Therefore A is a switcher, meaning B's statement is false and B is a liar. C's statement is true, so they are a saint.

ii. There is only one liar, meaning just one of the three statements can be true. Therefore the contrarian must lie, making them either the first to speak or after the saint. Both P and Q (the first two statements) say that the speaker after them is the liar. Therefore the contrarian cannot come after the saint (the contrarian is not the liar) and must come first. P is the contrarian and lies, meaning Q is the saint and R is the liar.

iii. If Y is the liar:
X's first statement is true and since they are the first to speak, X cannot be a contrarian. Therefore X is a switcher and Z is the contrarian.
Y's first statement is false (Z is a contrarian), consistent with Y being a liar.
Z's statement is a lie. However, the previous statement was a lie and a contrarian cannot lie if the preceding speaker lied.
∴ Y is not the liar.

If Z is the liar:
X's first statement is a lie, meaning they could be a switcher or a contrarian.
Y's first statement is true, meaning they could be a switcher or a contrarian.
Z is lying, consistent with Z being the liar.
X is lying again in their second statement, meaning they are the switcher (contrarians tell the truth after a lie)
Y must be a contrarian. However the last statement is a lie and a contrarian would tell the truth, so it is impossible.
∴ Z is not the liar.

If X is the liar:
X's first statement is a lie, as Y is not the liar.
Y's first statement is a lie, meaning Y is the switcher (contrarians tell the truth after a lie). Z is the contrarian.
Z's statement is true, X's statement is a lie and Y's last statement is true, all of which are consistent.
∴ X is the liar.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Ox-words are sequences of letters a and b that are constructed according to the following rules:

I. The sequence consisting of no letters is an Ox-word.

II. If the sequence W is an Ox-word, then the sequence that begins with a, followed by W and ending in b, written aWb, is an Ox-word.

III. If the sequences U and V are Ox-words, then the sequence U followed by V, written UV, is an Ox-word.

All Ox-words are constructed using these rules. The *length* of an Ox-word is the number of letters that occur in it. For example $aabb$ and $abab$ are Ox-words of length 4.

(i) Show that every Ox-word has an even length.

(ii) List all Ox-words of length 6.

(iii) Let W be an Ox-word. Is the number of occurrences of a in W necessarily equal to the number of occurrences of b in W? Justify your answer.

You may now assume that every Ox-word (of positive length) can be written *uniquely* in the form $aWbW'$ where W and W' are Ox-words.

(iv) For $n \geqslant 0$, let C_n be the number of Ox-words of length $2n$. Find an expression for C_{n+1} in terms of $C_0, C_1, \cdots, C_n$. Explain your reasoning.

7i. A sequence with no letters (I) is even. II adds two letters at a time to a word, meaning the total will stay even. III combines previous ox-words, all of which are even. Therefore all words using III will be even and all ox-words have an even length.

ii. aaabbb
ababab
abaabb
aabbab
aababb

iii. Yes. I has 0 "a"s and 0 "b"s. II adds 1 of each, meaning all words formed by this rule will have equal numbers of "a"s and "b"s. III combines previous ox-words, all of which have equal "a"s and "b"s, $\therefore$ all ox-words will have an equal balance between "a"s and "b"s.

iv. C_{n+1} is the number of ox-words of length $2n+2$.
If $awbw'$ is of the length $2n+2$, the length of w + the length of $w' = 2n$.
Let the length of w be $2k$ (where $k \leq n$). There are C_k Ox-words of length $2k$.
The length of w' is $2(n-k)$. There are C_{n-k} Ox-words of this length.
Therefore, the number of ox-words of length $2n+2$ is $\sum_{k=0}^{n} C_k C_{n-k}$ (for each value of k from 0 to n)

$$\therefore C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$